Using the parametric space to assess the robustness of solutions to MOLP problems with interval coefficients

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Abstract

Mathematical models for decision support in complex management and engineering problems require both the explicit consideration of multiple, often conflicting, axes of evaluation of the merits of potential alternative solutions and the incorporation of elements dealing with the uncertain nature of the coefficients. Capturing this uncertainty through intervals is an interesting approach to model uncertainty because it does not impose stringent applicability conditions and enables the development of techniques well suited for an interactive application. This paper proposes an interactive approach to tackle multiple objective linear programming (MOLP) models with interval coefficients, which is based on the analysis of indifference regions of non-dominated solutions in the parametric space as a means to assess their robustness in the sense of "immunity" to unknown but bounded coefficients.

Keywords: Uncertainty modeling, Multiobjective linear programming; Interval coefficients.

1. Introduction

The main argument for justifying the need to consider models and methods for decision support taking explicitly into account multiple evaluations aspects in multiobjective mathematical programming is generally of "realistic" type. That is, the assessment of the merits of alternative solutions in real-world problems inherently involves multiple evaluation axis and models are required to reflect this reality as accurately as possible. Therefore, mathematical models for decision support become more representative of the actual decision situation whenever multiple objective functions, which are typically conflicting and incommensurable, are formulated explicitly, instead of aggregating evaluation aspects of distinct nature in a single function (usually a cost or benefit indicator) by transforming their impacts into monetary units. However, the relevance of the multiobjective approach goes beyond this "realistic" argument and it holds an intrinsic value-added role in the model building and the result analysis processes, supporting reflection and creativity in face of a larger universe of potential solutions since an optimal solution no longer exists (Roy, 1990; Bouyssou, 1993).

Since a prominent optimal solution does not exist in multiple objective programming models, methods strive for the computation of non-dominated (efficient) solutions. These are the feasible solutions for which no other feasible solution exists strictly improving all objective function values; that is, the improvement of an objective function value can be obtained only by accepting to degrade at least another objective function value.

Multiobjective models and methods provide the decision makers (DM) a framework for making more rational the comparisons within the (usually large) set of non-dominated solutions, offering a clearer perception of the conflicting aspects under evaluation and the capability to grasp the tradeoffs that must be made for the selection of a balanced solution, which can be accepted as the final satisfactory compromise solution to the decision process. In this framework the DM's preferences play an important role, being understood as the construct the DM leans on for evaluating and eventually selecting that solution from the non-dominated solution set.

However, this preference structure is seldom clearly shaped. Therefore, the need arises to offer the DM a flexible decision aid environment through which he/she can experiment distinct search paths (giving privilege to one or another objective function, grasping trade-offs between the functions in different parts of the search space, etc.). That is, an operational decision support framework is required capable of offering an interactive way of progressing in a selective way in acquiring and processing the information associated with newly computed solutions (rather than imposing a rigid and pre-specified sequence of computation and information exchange steps). In this context, the interactive process is understood as a learning process in which the DM can go through the non-dominated region in a progressive and selective way by using the information gathered so far to express new preference information to guide the ongoing search for new solutions. The DM intervenes in the solution search process by inputting information into the procedure, which in turn is used to guide the computation phase towards solutions that more closely correspond to his/her (evolutionary) preferences.

Uncertainty is an intrinsic characteristic of real-world problems arising from multiple sources of distinct nature. Uncertainty emerges from the ever-increasing complexity of interactions within social, economical and technical systems, characterized by a fast pace of technological evolution, changes in market structures and new societal concerns. In a multiobjective setting, the elicitation of the DM's preferences contributes to add a new uncertainty dimension to the decision-aid process.

In this context of decision problems characterized by model and preference uncertainties, it is important to provide the DM with information enabling him/her to select robust solutions. The concept of robust solution broadly refers to some kind of "immunity" to data uncertainty (whatever it happens, the solution is good in most conditions and it is not very bad in none) or to an adaptive capability regarding an uncertain future (flexibility associated with keeping as many as possible options open given a decision previously made), guaranteeing an acceptable performance even under changing conditions (drifting from "nominal data").

Therefore, decision-aid models and, in particular, mathematical programming models must be able to capture both essential features of real-world problems: uncertainty and multiple objective functions. Methods must then be designed to tackle in a creative way these issues, in the operational framework of decision support tools.

In the following sections, the main sources and types of uncertainty in the multiple objective mathematical programming models are briefly reviewed as well as the need to assess the robustness of the potential solutions. Interactive techniques based on the exploitation of the parametric diagram for three-objective linear programming problems are proposed, in which the uncertainty associated with the model coefficients is modeled by means of intervals.

2. Uncertainty in multiple objective mathematical programming models

Since real-world problems are generally very complex, it is practically impossible that decision aid models, and mathematical programming models in particular, could capture all the relevant inter-related phenomena at stake, get through all the necessary information, and also account for the changes and/or hesitations associated with the DM's preferences. For instance, non-linear relations the functional form of which is unknown can be made linear for the sake of tractability, since linear programming models are easier to tackle and other approaches may be equally disputable. This structural uncertainty is associated with the global knowledge about the system being modeled. Moreover, input data used to develop the model coefficients may suffer from imprecision, incompleteness or be subject to changes.

The term uncertainty is used herein with respect to situations in which the potential outcomes cannot be described by using objectively known probability distributions, nor can they be estimated by subjective probabilities. In this sense, uncertainty is distinct from risk, this term referring to a situation in which the potential outcomes can be described in reasonably well-known probability distributions (Haimes, 2004). Therefore, uncertainty encompasses situations characterized by parameters whose values: are not known precisely (or only rough estimates are available), result from statistical data or measurement tools, are arbitrary, incomplete, not credible, contradictory (according to different sources) or controversial (according to different stakeholders), reflect the DM's preference structure and values (which can evolve as more knowledge is acquired throughout the decision process or are difficult to elicit explicitly). The term parameter is herein used in a broad sense, encompassing both model coefficients and other technical devices required by the decision support methodology such as weights, thresholds, aspiration levels, reservation levels, etc.

On one hand, the explicit consideration of multiple objective functions contributes to make models more adequate reflecting (a broader view of) reality and the need to weigh trade-offs in the search for a compromise solution. On the other hand, it adds a new uncertainty dimension since the DM's preferences are required and used in the decision-aid process. These preferences are often unclear, ambiguous and unstructured. This issue gains more importance in a context in which it is unworkable to compute all non-dominated solutions and the DM's preferences play a key role in guiding the search.

Therefore, it is necessary to provide the DM methodologies and computer tools, which can assist him/her in assessing the robustness of solutions regarding the uncertainty, arising from several sources and of different types, underlying the decision process. In this way the interactive decision process can capture the changes in the input data (studying different discrete scenarios or the evolution of a given scenario), the redefinition of the model (incorporating new elements of reality through the consideration of new decision variables, constraints and/or objective functions), and also the evolutionary nature of his/her preferences (testing, for example, his/her judgments that reveal more influent in guiding the interactive search process towards certain regions of the non-dominated solution set). Having in mind the constructive nature of the decision support process, a methodological and operational framework is proposed that enables to take into account the uncertainties associated with imprecisely known model coefficients and the elicitation of preferences.

Interval programming is an interesting approach to model uncertainty regarding the coefficients of mathematical programming models, mainly because it does not impose stringent applicability conditions. The underlying assumption is that the actual coefficients are not generally known with precision. They derive from estimates by experts, subjective judgments in complex environments, imprecise measurements, etc. However, it is possible in most situations to specify with a reasonable degree of accuracy ranges of admissible values for the coefficients, but it is difficult to state a reliable probability distribution for this variation. That is, each coefficient is a closed interval rather than a single real value (a region the coefficients can possibly take). An illustrated overview of interval programming in MOLP models can be seen in Oliveira and Antunes (2007).

Other approaches to model uncertainty in decision support models involve the construction of scenarios, stochastic programming, fuzzy programming, sensitivity analysis, etc.

In the context of mathematical programming models, scenarios (embodying different sets of assumptions of plausible future states) can be made operational by the specification of coefficients (for instance, within intervals) for each scenario. This generally leads to a high number of scenarios (due, for instance, to the possible combinations resulting from the simultaneous and independent variations of coefficients) and it is necessary to design a form of pruning or aggregating distinct “patterns”. In this case it is expected that solutions selected as potential outcomes of the decision process are robust regarding the plausible conditions in which the system can be encountered in the future (that is, across scenarios).

Sensitivity analyses are well-known techniques in mathematical programming providing information on the behavior of optimal solutions (in single objective optimization) and the range of variation of the model coefficients such that the optimal solution is maintained. More precisely, for example in linear programming, sensitivity analysis computes the ranges in which the original model coefficients (or some perturbation parameters) can change such that the optimal basis remains optimal for the "perturbed" problem. This concept cannot be translated in a straightforward way to a MOLP context since several non-dominated solutions exist (even if only basic solutions are considered) and the DM's preferences also play a role (that is, to analyze all solutions would not be an worthwhile effort since most of them would be of no interest for the DM). Furthermore, sensitivity analysis is a "post-identification" technique, in the sense that it enables to analyze the behavior of a given (optimal or non-dominated) solution after it is computed, but it is not of help to be integrated in the search process to generate robust solutions.

The main concept in stochastic programming is the one of recourse, in the sense of a capability to make corrective measures in face of a random event. A typical approach in stochastic programming consists in defining scenarios (conditions that can be identified and taken as representative of the state in which the system can be identified in the future) to which probabilities are assigned based on postulated or empirically verified distributions. A formulation of a linear stochastic programming with two periods consists in the minimization of the cost associated with the decision to be made in the first period (before the realization of uncertain coefficients) plus the expected cost of the recourse decision in the second period. The decision variable values in the second period may be interpreted as the operational recourse (corrective) measures against the infeasibilities arising from a particular realization of uncertainty.

The modeling of data uncertainty can also be made by using concepts of fuzzy set theory. Initially, the use of the fuzzy set theory in the framework of mathematical programming problems aimed at making less rigid the notion of constraint by giving the same nature to objective functions and constraints and making flexible (in the sense of gradual) the inequality, or equality, between both sides of the constraints and objective functions (in this case requiring the specification of a desired level to be attained). It then evolved in a sense similar to stochastic programming to model the imprecise nature of the coefficients of mathematical programming models by using possibilistic distributions (assigning a membership function to the fuzzy model coefficients). As a set, an interval is a fuzzy set with a rectangular membership function.

3. Robust solutions in MOLP using interval programming

It must be recognized that DMs do not expect from the decision support process (generally mediated by an analyst with technical and methodological knowledge and sensible towards the problem at hand) a "ready-to-use" solution but rather help in a process of gathering, in a constructive manner, information which can be used to make well-informed decisions, acting as anchors to the selection of a course of action or just to pave the way for further reflection about the problem and also his/her own preferences. Multiple objective models play a value-added role by widening the spectrum of potential outcomes (that is, a true decision problem is at stake and not just the decision on accepting or rejecting an "optimal" solution). Therefore, it is necessary that the potential solutions could be compatible, in the sense of reachable, to a set of acceptable combinations for the input values.

In the context of optimization problems Mulvey et al. (1995) understand robustness in the sense of closeness to feasibility and to optimality across the scenario universe. The aim is to compute solutions that are fairly insensitive to any scenario realization. Two robustness measures are defined in Kouvelis and Yu (1997) in the context of discrete optimization problems: absolute robustness - the worst-case performance (minimax); and the robust deviation - worst case performance difference between the given solution and the best solution (minimax regret). These measures are conservative (pessimistic) ones. Bertsimas and Sim (2004) proposed a robust approach to linear programming problems with uncertain data, adjusting the levels of conservatism of robust solutions in terms of probabilistic bounds of constraint violations. Vincke (1999) proposed an operational formalism to define the concepts of robust solutions and robust methods in decision support.

Let us consider the multiple objective linear programming (MOLP) problem

$$\begin{array}{ll} \text{"max"} & \mathbf{f}(\mathbf{x}) = C\,\mathbf{x} \\ \text{s. t.} & \mathbf{x} \in X \\ & X = \{\, \mathbf{x} \in \mathbb{R}^n \mid A\,\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0},\ \mathbf{b} \in \mathbb{R}^m \,\} \end{array}$$

All model coefficients (the objective function matrix C, the technological matrix A, and the right-hand side vector **b**) are closed intervals defined by (an ordered pair) lower (left) and upper (right) bounds of the coefficient. The interval may also be denoted by its center (typically the nominal values) and width (which can be defined by a percentage variation around the nominal value). An interval vector is a vector whose components are interval numbers. A precisely known coefficient has equal left and right limits.

It is well known that non-dominated basic solutions to the MOLP problem can be obtained by optimizing a weighted-sum scalarizing function:

$$\max \ \lambda_1 f_1(\mathbf{x}) + \lambda_2 f_2(\mathbf{x}) + \ldots + \lambda_p f_p(\mathbf{x})$$

$$\text{s. t. } \mathbf{x} \in X$$

$$\boldsymbol{\lambda} \in \Lambda \equiv \{\boldsymbol{\lambda} : \boldsymbol{\lambda} \in \Re^p, \textstyle\sum_{k=1}^{p} \lambda_k = 1, \lambda_k \geq 0, k=1,2,...,p\}$$

For computational purposes $\lambda_k > 0$, in order to avoid weakly non-dominated solutions.

A weighting vector $\boldsymbol{\lambda}=(\lambda_1, \lambda_2,\ldots, \lambda_p)$ can be represented as a point on the parametric diagram Λ. This is a geometrical (p-1)-dimensional simplex in a p-dimensional Euclidean weight space (p being the number of objective functions). This parametric diagram is especially interesting for displaying useful information to the DM in problems with three objective functions (p=3). The parametric diagram performs the role of a consistent means in which it is graphically displayed information regarding the solutions computed and used by the DM/analyst to provide indications about the regions in which the search shall proceed. The aim is to provide the DM information in a way that supports the emergence of insights in the progressive search for potential solutions to the problem (Clímaco and Antunes; 1989; Antunes and Clímaco, 1992).

The decomposition of the parametric diagram is used as an operational means to convey information to the DM. The graphical display (for p=3) of the set of weights that leads to each non-dominated (basic) solution can be achieved through the decomposition of the parametric diagram. From the simplex tableau corresponding to a non-dominated basic solution to the weighted-sum problem, the corresponding set of weights is given by $\boldsymbol{\lambda}^T W \geq \mathbf{0}$, where $W=C_B B^{-1}{}_N - C_N$ is the reduced cost matrix. B (C_B) and N (C_N) are the sub-matrices of A (C) corresponding to the basic and non-basic variables, respectively.

The region comprising the set of weights corresponding to a non-dominated basic solution k, defined by $\Lambda^k \equiv \{\boldsymbol{\lambda}^T W \geq \mathbf{0}, \boldsymbol{\lambda} \in \Lambda\}$, is called indifference region. The DM can thus be indifferent to all the combinations of weights within this region, because they lead to the same non-dominated solution. The boundaries between two contiguous indifference regions represent the non-basic efficient variables (those that when introduced into the basis lead to an adjacent non-dominated vertex through a non-dominated edge). A common boundary between two indifference regions means that the corresponding non-dominated solutions are connected by a non-dominated edge. If a point $\boldsymbol{\lambda}$ belongs to several indifference regions this means that they correspond to non-dominated solutions lying on the same face (Clímaco and Antunes; 1989; Antunes and Clímaco, 1992).

The analysis of the parametric diagram is thus a valuable decision aid tool in "learning" the shape of the non-dominated solution set, and consequently in grasping the potential solutions to the MOLP problem. The decomposition of the parametric diagram as a means to make a progressive and selective learning of the non-dominated solution set and evaluate the stability of selected non-dominated solutions to changes in the coefficients is exploited in (Clímaco and Antunes; 1989; Antunes and Clímaco, 1992; Antunes and Clímaco, 2000).

The process begins by offering the DM the possibility of freely compute non-dominated solutions using the nominal values (all coefficients in the midpoint of their intervals). In particular, the non-dominated solutions that individually optimize each objective function are computed as well as some well-dispersed solutions with the aim of having a first overview of the non-dominated solution set (figs. 1a-1b). Due to the small size of this example (3 objective functions, 4 decision variables and 4 constraints) we opted to compute all basic (vertex) non-dominated solutions for illustrative purposes only. It must be noticed that this approach is generally intended to avoid an exhaustive search by recognizing that the knowledge about the solutions computed so far (and, in particular, their objective function values) enables to decide whether it is still necessary to compute solutions using weight sets belonging to certain regions of the parametric diagram not yet filled.

In this situation 8 vertex non-dominated solutions exist (note that solutions 2 and 8 are alternative optima to objective function $f_2(\mathbf{x})$), defining 3 non-dominated faces (1-5-6, 5-6-4-7-2, and 7-8-2; note that solutions lying on the face defined by solutions 3-4-7 are dominated by the solutions on the non-dominated edges 3-4- and 4-7); see also table 1.

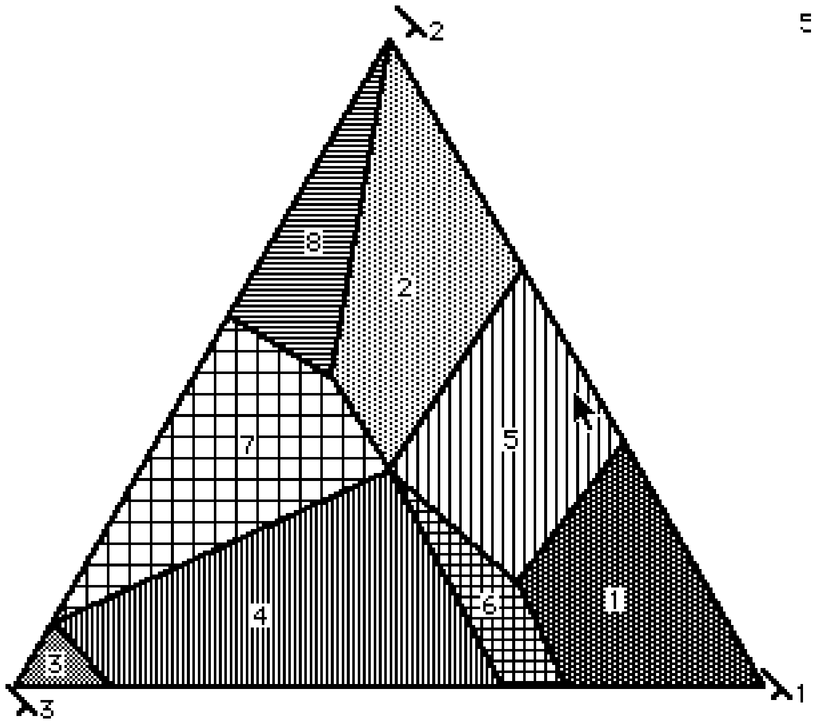

Figure 1: Decomposition of the parametric diagram with all coefficients in the midpoint of the intervals (all figures are screen copies generated by the TRIMAP package)

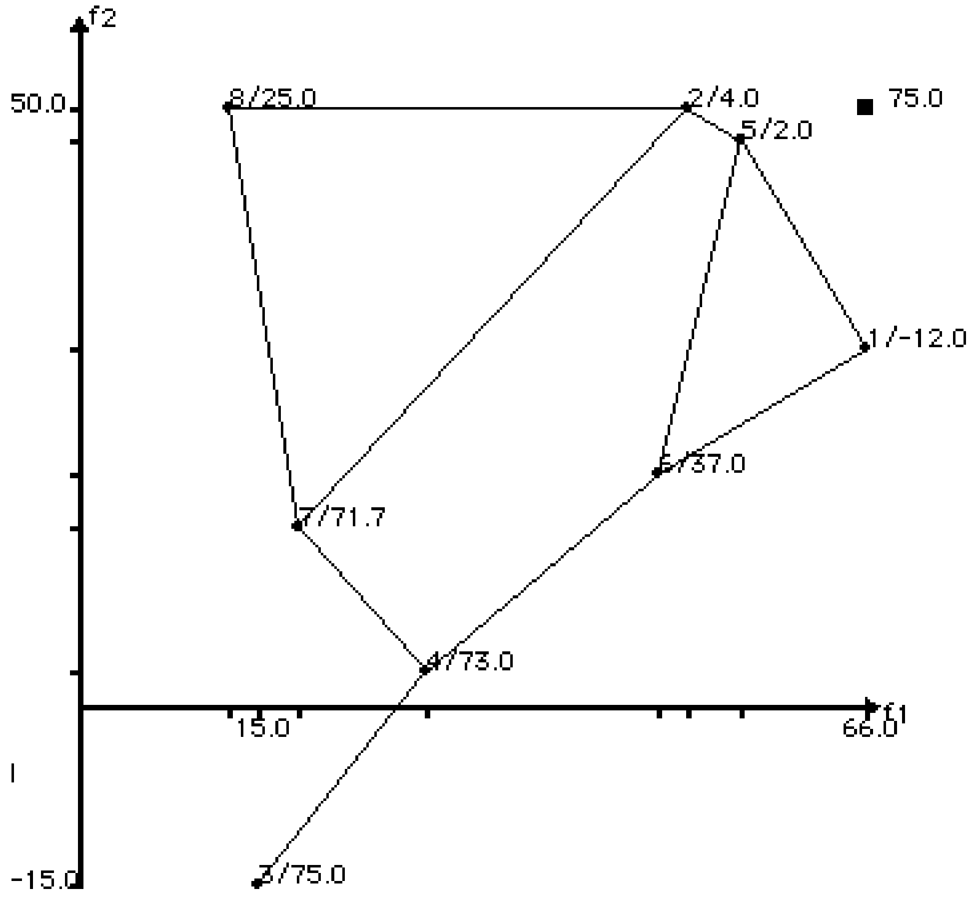

Figure 1b: Projection of the objective function space corresponding to the decomposition of the parametric diagram in fig. 1a

(for instance, in 8/25.0, 8 denotes the identification of the solution and 25.0 is the value of f_3)

Table 1 – Basic (vertex) non-dominated solutions with all the coefficients in the midpoint of the intervals (in the column of the basic variables $\mathbf{x}_B$, s_i stands for the slack associated with constraint i)

Solution	f_1	f_2	f_3	$\mathbf{x}_B$
1	66	30	-12	x1, x3, s3
2	51	50	4	x1, x4, s1
3	15	-15	75	x2, s1, s3
4	29	3	73	x2, x3, s1
5	55.5	47.5	2	x1, x3, x4
6	48.5	19.5	37	x1, x2, x3
7	18.333	15	71.667	x2, x4, s1
8	12.5	50	25	x4, s1, s2

The uncertainty associated with the model coefficients is then taken into account by computing solutions using coefficients that are randomly generated within their intervals. The indifference regions associated with all (vertex) non-dominated solutions already computed are the starting point. Weighted-sum scalarizing functions are then constructed using random coefficients within the intervals and the weight vectors are also randomly generated within the indifference region for the “nominal” (midpoint coefficients) situation (the one described in figs. 1a-b and table 1). As a result of this simulation, the robustness of the solution regarding coefficient changes is assessed taking into account both the frequency the same basic non-dominated solution (optimal basis to the scalarizing problem) is obtained and the degree of superposition of the corresponding indifference regions in the parametric diagram.

Using the interactive visual feedback provided by the parametric diagram the changes regarding the “nominal” (midpoint) coefficients can be analyzed, for an instantiation of the interval coefficients and the weights used in the scalarizing problem, as displayed in figs. 2a-b and table 2.

Note that with these particular random coefficients the former solutions 2, 4, 6 and 8 (in fig. 1) are no longer non-dominated (they do not have a corresponding indifference region in the parametric diagram), which can be perceived as an indication of being less robust regarding coefficient changes. In this particular coefficient realization, 7 vertex non-dominated solutions and 3 non-dominated faces (1-6-4-5, 2-4-5, and 3-7-4-6) exist. Other conclusions can be drawn, such as the non-dominated solutions that optimize individually each objective function (solutions 1, 2 and 3 in figs. 2a-b and table 2) are now unique (i.e., no alternative solutions exist for any objective function). For the situations displayed in figs. 1

and 2, the degree of superposition is 1 for the solutions 1 and 3, but is less than 1 for solutions 5 and 7.

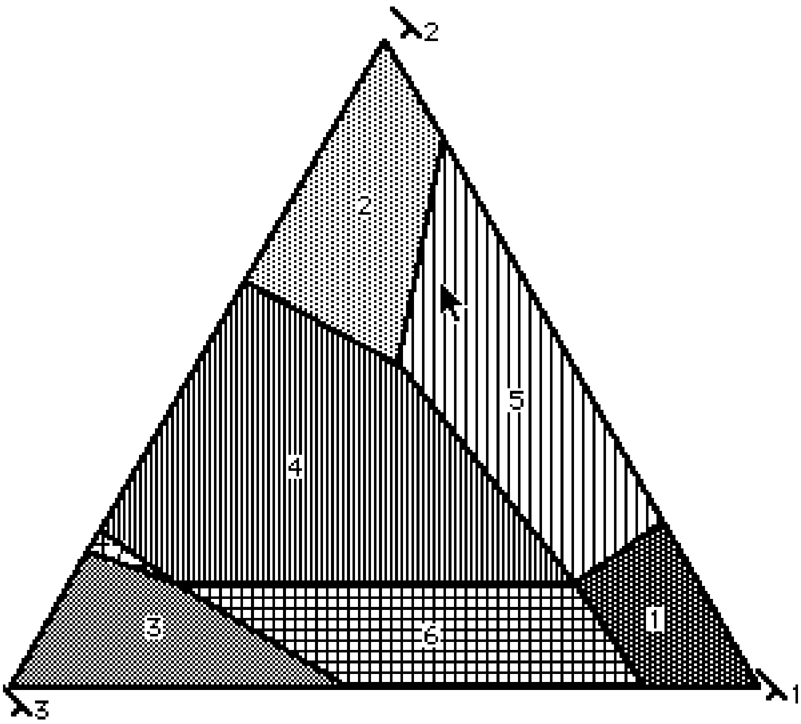

Figure 2a: Decomposition of the parametric diagram (random coefficients within of the intervals)

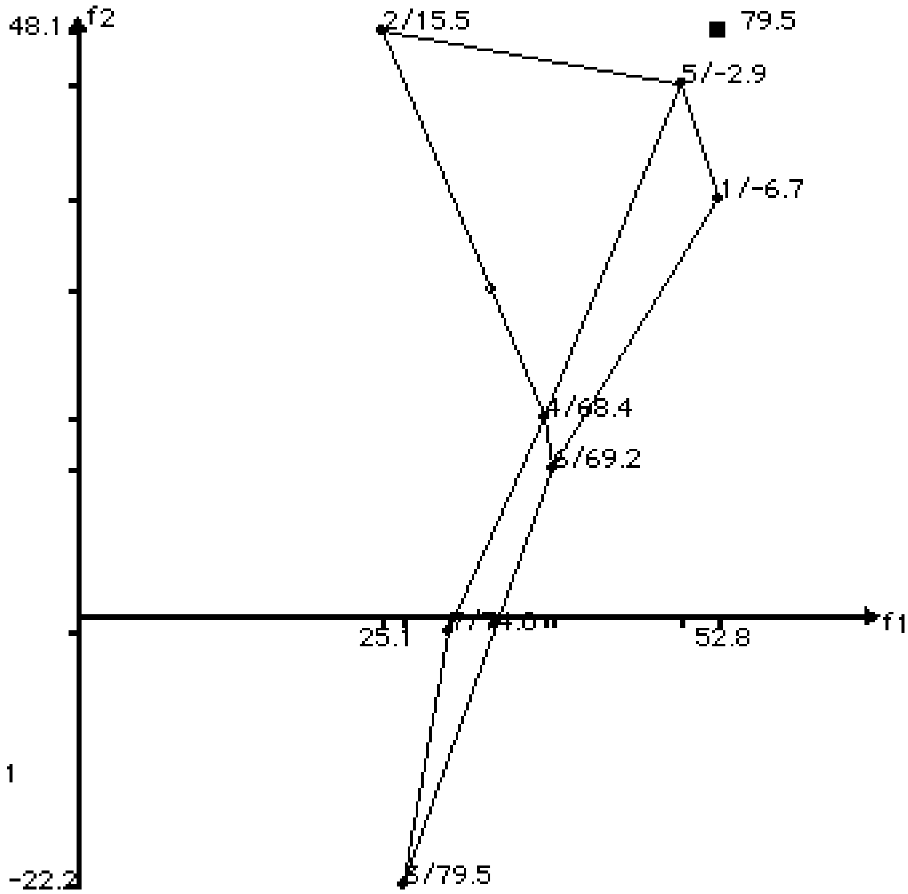

Figure 2b: Projection of the objective function space corresponding to the decomposition of the parametric diagram in fig. 2a

Table 2 – Basic (vertex) non-dominated solutions with random coefficients within the intervals and random weights within the "nominal" indifference regions

Solution	f_1	f_2	f_3	$\mathbf{x}_B$	Same as in table 1
1	52.765	34.335	-6.658	x1, x3, s3	1
2	25.120	48.095	15.490	x3, x4, s2	---
3	26.808	-22.181	79.509	x2, s1, s3	3
4	38.367	16.256	68.353	x2, x3, x4	---
5	49.563	43.646	-2.884	x1, x3, x4	5
6	39.195	12.005	69.163	x2, x3, s3	---
7	30.549	-1.310	73.962	x2, x4, s1	7

This type of analysis can be done for the most "favorable"/"unfavorable" situations, that is with the coefficients in the interval endpoints that favor/disfavor the best/worst objective functions values (matrix C coefficients in their upper/lower values in functions to be maximized, and matrix A coefficients in their lower/upper values and vector **b** coefficients in their upper/lower values in ≤ constraints to "enlarge"/"shrink" the feasible region).

Another possibility to assess the solution robustness is to consider the centroid of the "nominal" indifference region (fig. 1a) as the most stable set of weights leading to that solution. Then, this weight vector would be constant (not requiring random generation of weights) and the associated solution would be assigned a robustness degree based just on the frequency the same basic solution is obtained by optimizing a weighted-sum scalarizing function with that set of weights in a simulation run using random coefficients with the intervals.

This approach based on randomly generated coefficients within the intervals (as well as weight vectors for the scalarizing problems to be solved) prevents the drawbacks of other approaches that are based on the extreme rays of the objective functions. In this case, if the gradients of the objective functions are highly correlated, the scope of the search may be reduced and the number and diversity of solutions that can be computed is impaired.

4. Conclusions

Interval programming is an interesting approach to model uncertainty because it just requires the specification of acceptable lower and upper bounds for the coefficients (that is, these are unknown but bounded). In this paper some possibilities have been suggested for using the information associated with indifference regions in the parametric space (related to basic non-dominated solutions) and random coefficients generated within intervals (representing the uncertainty underlying model coefficients in MOLP problems) to assess the robustness of non-dominated solutions.

As the models coefficients change within the intervals specified, solutions can become non-dominated or even infeasible. The robustness of each non-dominated solution is appraised regarding the nominal situation (interval midpoint coefficients) based on the frequency of appearance of non-dominated solutions, also taking into account the degree of super-position between the nominal indifference regions and the indifference regions resulting from randomly generated coefficients within the intervals.

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